

Existence of log canonical closures

Minimal Model Program Learning Seminar

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Main Proposition

Proposition 1 (3.1)

Let (X, Δ) be a dslt pair, projective over a normal variety U . And $n : X^n \rightarrow X$ be the normalization. Write $n^*(K_X + \Delta) = K_{X^n} + \Delta^n + \Gamma$. Where Γ is the double-locus. Assume that:

- There exists an open set $U^0 \subset U$, such that if we write $(X^0, \Delta^0) = (X, \Delta) \times_U U^0$, then $K_{X^0} + \Delta^0$ is semi-ample over U^0 .
- The image of any non-klt center of $(X^n, \Delta^n + \Gamma)$ intersects U^0 , and
- $K_{X^n} + \Delta^n + \Gamma$ is semi-ample over U^0 .

Then, $K_X + \Delta$ is semi-ample over U .

Definition 1 (3.2)

Let X be a scheme. A *stratification* of X is a decomposition of X into a finite disjoint union of reduced locally closed sub-schemes

We write $X = \bigcup_i S_i X$ where $S_i X \subset X$ is the i -th dimensional stratum. Such a stratified scheme is denoted by (X, S_*) . We assume that $\bigcup_{i \leq j} S_i X$ is closed for every j .

The boundary of (X, S_*) is the closed subscheme

$$\underline{BX} := \bigcup_{i < \dim X} S_i X = X \setminus S_{\dim X} X.$$

Let (X, S_*) , (Y, S_*) be stratified schemes. We say that $f : X \rightarrow Y$ is a stratified morphism if $f(S_i X) \subset S_i Y$ for every i .

$\bigcup_{i \leq j} S_i X =: S_j^* X$ is also closed

Example 2

Let (X, Δ) be a log canonical pair. Let $S_i^* \subset X$ be the union of all non-klt centers of (X, Δ) of dimension $\leq i$, and

$S_i X := \overline{S_i^*(X, \Delta) \setminus S_{i-1}^*(X, \Delta)}$. We call this the *lc stratification* of (X, Δ) .

(X, Δ)

Definition 3 (3.4)

(N) We say that (X, S_*) has *normal strata* if each $S_i X$ is normal

(SN) We say that (X, S_*) has *seminormal boundary* if X and the boundary BX are both seminormal

(HN) We say that (X, S_*) has *hereditary normal boundary* if:

- X satisfies (N)
- The normalization $\pi : X^n \rightarrow X$ is stratifiable, and
- $B(X^n)$ satisfies (HN)

✓
Sat. HN
HSN

(HSN) We say that (X, S_*) has *hereditary seminormal boundary* if:

- X satisfies (SN)
- The normalization $\pi : X^n \rightarrow X$ is stratifiable, and
- $B(X^n)$ satisfies (HSN)

We use HN and HSN to get
Quotients by finite relations

The lc stratification satisfies (N), (SN), (HN) and (HSN).

Example 4

Take

$$X = (x^2 = y^2(y + z^2)) \subset (A)^3$$

. With $S_1 = (x = y = 0)$. Then S_1 and S_2 are smooth. The normalization of X is:

$$X^n = (x_1^2 = y + z^2) \subset \mathbb{A}^3.$$

$$\lambda_1 = \frac{x}{y}$$

And the preimage of $S_1 X$ is $(y = x_1^2 - z^2 = 0)$, which is not normal.

So it has (N) but not (HN).

Definition 5 (3.6)

Let Y be a normal scheme. A *minimal qlc structure* on Y is a proper surjective morphism $f : \underline{(X, \Delta)} \rightarrow Y$, where:

- (X, Δ) is a log canonical pair
- $\mathcal{O}_Y = f_* \mathcal{O}_X$, and
- $K_X + \Delta \sim_{f, \mathbb{Q}} 0$

quasi log canonical.

Definition 6 (3.7)

Let $f : (X, \Delta) \rightarrow Y$ be a minimal qlc structure. We define the f -qlc stratification $(Y, S_*(X/Y, \Delta))$ in the following way. Let \mathcal{H}_X denote the set of all non-klt centers of (X, Δ) . For each $\underline{Z} \in \mathcal{H}_X$, let:

$$W_Z = f(Z) \setminus \bigcup_{Z' \in (H)_X, f(Z) \not\subseteq f(Z')} f(Z').$$

Then $\bigcup_{Z \in \mathcal{H}_X} W_Z$ is the qlc -stratification.

$\bigcup W_Z$ of given dimension

give the stratification

qlc strat. satisfies (HN) and (HSN)

Definition 7

Let X and R be U -schemes. A pair of morphisms $\sigma_1, \sigma_2 : \underline{R} \rightrightarrows X$ is called a pre-relation. It is called *finite* if both morphisms are finite and a relation if $\sigma : R \rightarrow X \times_U X$ is a closed embedding.

Definition 8 (3.3)

Let $(X < S_*)$ be a stratified scheme. A relation $\sigma : R \rightrightarrows X$ is *stratified* if each σ_i is stratifiable and $\sigma_1^{-1}S_i = \sigma_2^{-1}S_i$.

pre-relation \rightsquigarrow relation ✓✓

Definition 9

Let X and R be reduced U -schemes. We say that a relation $\sigma : R \rightrightarrows X$ is a *set theoretic equivalence relation* if:

- σ is geometrically injective.
- R contains the diagonal. *(reflexive)*
- There is an involution τ_R on R , such that $\tau_{X \times X} \circ \sigma \circ \tau_R = \sigma$. *(symmetric)*

- For $1 \leq i < j \leq 3$. Taking $X_i := X$ and $R_{ij} \subset X_i \times_U X_j$. Then the coordinate projection of $\text{red}(R_{12} \times_{X_2} R_{23})$ to $X_1 \times_U X_3$ factors:

$$\text{red}(R_{12} \times_{X_2} R_{23}) \rightarrow \underbrace{R_{13}}_{R_{13}} \rightarrow X_1 \times X_3.$$

f : no
pre-rel, eq.

transitive.

pro-f.
rel, eq.

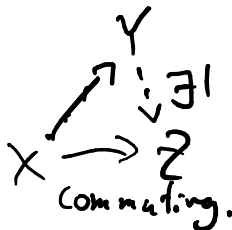
Definition 10

Let $\sigma : R \rightrightarrows X$ be a set theoretic equivalence relation. We say that $q : \underline{X} \rightarrow Y$ is a *geometric quotient* of X by R if:

- $\underline{q \circ \sigma_1} = \underline{q \circ \sigma_2}$.
- $q : X \rightarrow Y$ is universal with this property.
- $q : X \rightarrow Y$ is finite.

The geometric quotient is denoted by X/R .

$$Y = X/R$$



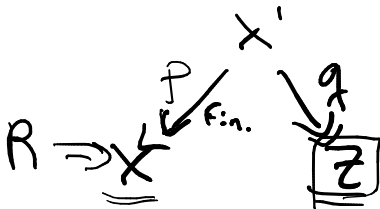
Lemma 11

Let $R \rightrightarrows X$ be a finite, set theoretic equivalence relation with X, R reduced and over a field of characteristic 0. let $\pi : X' \rightarrow X$ and $q' : X' \rightarrow Z$ be finite surjections, with either

- X, Z are semi-normal and the geometric fibers of q' are exactly the pre-images of R -equivalence classes.

• X, Z normal and such that the $\sigma_i : R \rightarrow X$ are open and over a dense subset of Z , the geometric fibers of q' are exactly the pre-images of R -equivalence classes.

Then $Z = X/R$



Theorem 12 (3.8)

Let (X, S_*) be a stratified excellent scheme or algebraic space over a field of characteristic 0. Assume that (X, S_*) satisfies (HN) and (HSN). Let $R \rightrightarrows X$ be a finite, stratified, set theoretic equivalence relation. Then:

- The geometric quotient X/R exists.
- $\pi : X \rightarrow X/R$ is stratifiable, and
- $(X/R, \pi_* S_*)$ also satisfies (HN) and (HSN).

Proof

Lemma 13

Let (X) be an excellent scheme over a field of characteristic 0 that is normal and of pure dimension d . Let $R \rightrightarrows X$ be a finite, set theoretic equivalence relation. Let $\underline{R^d} \subset R$ denote the d -dimensional part of R . Then

- $R^d \rightrightarrows X$ is a finite, set theoretic equivalence relation,
- The geometric quotient $\boxed{X/R^d}$ exists, and
- X/R^d is normal.

Proof

Idea is to get some
 $S \leadsto "X"$ and $"X"/S \cong X/R^d$

Proof of Lemma 13

$R^d \rightrightarrows X$ reflexive, symmetry. Follow from $R \rightrightarrows X$.
Transitive property does not

$\sigma_i: R^d \rightarrow X$ Finite morphisms \rightarrow image is normal, and both pure dimension

$\Rightarrow \sigma_i$ is universally open.

$R^d \times_x R^d \rightarrow R^d$ is open.

has pure dimension d .

Therefore, $R^d \times_x R^d \rightarrow R$ lies in the d -dim part R .

$$R^d \times R^d \rightarrow R \times R \rightarrow R \rightarrow X \times X$$

$$\downarrow$$

$$R^d \times R^d \rightarrow R^d \subseteq R \rightarrow X \times X$$

Splitting of projections.

Construct the quotient

we can assume X irreducible.

$X \times \dots \times X$ with $\pi_i: X \times \dots \times X \rightarrow X$
deg G_1 each coord proj.

And $R_{ij}, \Delta_{ij} = (\pi_i \times \pi_j)^{-1}(R)$
 $(\pi_i \times \pi_j)^{-1}(\Delta)$

Geometric points $\bigcap_{i,j} R_{ij} (x_1, \dots, x_m)$

s.t. any 2 are R-equivalent.

$\bigcap R_{ij} \setminus \bigcup \Delta_{ij}$ sequences (x_i, \dots, x_n)

giving entire equivalence classes.

$X' \xrightarrow{n} \overline{\bigcap R_{ij} \setminus \bigcup \Delta_{ij}}$

$S_m \curvearrowright X' \times \dots \times X$

$\curvearrowright \bigcap R_{ij} \setminus \bigcup \Delta_{ij}$

l.f.t $S_m \curvearrowright \boxed{X'}$

over a dense subset of X the S_m orbits are R-equivalence class,

$X' \rightarrow X$
 $X' \rightarrow X/S_m$

Criterion,
 $X'/S_m = X/R$.
quot. by fin. group. \exists .

$Y \xrightarrow{n} X/R^d$, X is normal

has to
comp with
 $R \rightrightarrows X \rightarrow X/R^d$
 $\searrow \exists!$
 W

l.f.t, l.f.t.s

\Rightarrow its the g. quotient

Proof of Theorem 3.8

$$R \rightrightarrows X$$

We use induction $d = \dim X$.

$$(X^n, S^n) \rightarrow (X^n, S_*)$$

$$\begin{array}{c} \uparrow \uparrow \\ R^n \\ \text{pullback.} \\ \text{keeps conditions on } R. \end{array}$$

$$\begin{array}{c} \uparrow \uparrow \\ R \end{array}$$

$$X^{nd} \subseteq X^n$$

d -dim. irr. comp.

$$X^{nd} \rightarrow \underline{X^{nd} / R^{nd}} \text{ exists and normal.}$$

$$X^{nl} := X \setminus X^{nd} \text{ the lower dim. part.}$$

$$R^{nd} \rightrightarrows X^n \text{ (by being trivial in } X^{nl})$$

$$X^n / R^{nd} = (X^{nd} / R^{nd}) \sqcup X^{nl}.$$

$$\Rightarrow B(X^n / R^{nd}) = \underbrace{B(X^{nd} / R^{nd})}_{\text{quotient in boundaries exist by induction}} \sqcup X^{nl}.$$

quotient in boundaries exist by induction.

$$\begin{array}{ccc} B(X^{nd} / R^{nd}) \sqcup X^{nl} = B(X^n / R^{nd}) & \hookrightarrow & X^n / R^{nd} \\ \downarrow & & \downarrow \\ B(X^n / R^{nd}) /_{\tilde{q} \circ R^{nl}} & \rightarrow & X^n /_{\tilde{q} \circ R^{nl}} \\ & & \downarrow \\ & & R^n \end{array}$$

Y will be the quotient X^n / R^n

□

Lemma 14 (3.9)

Let (X, S_*) be a stratified space satisfying (N) and $Z \subset X$ a closed subspace which does not contain any of the irreducible components of the $S_i X$. Let $R \rightrightarrows (X, S_*)$ be a pro-finite, stratified, set theoretic equivalence relation. If $R|_{X \setminus Z}$ is a finite set theoretic equivalence relation, then R is also a finite set theoretic equivalence relation.

Proof

pro-finite.

R is a union of finite relations.

We need to check that R has finitely many components.

Proof of Lemma 3.9

We can do this Strata by Strata.

We have normal.

Every irred comp of R dominates
an irreducible component of X .

Finiteness (over a dense subset) $X \setminus Z$
gives finitely many components
of R .

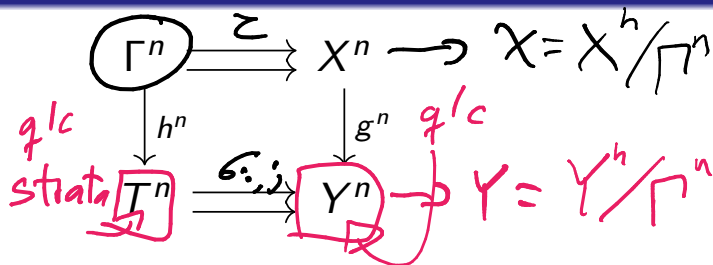
□

Let (X, Δ) be a dslt pair, Γ^n the normalization of the double locus of $\Gamma \subset X^n$ and $\tau : \Gamma^n \rightarrow \Gamma^n$ the induced involution. Then the relation $\tau_1, \tau_2 : \Gamma^n \rightrightarrows X^n$ has quotient given by the normalization:

$$\pi : \underbrace{X^n} \rightarrow \underline{X} = \underline{X^n / \Gamma^n}.$$

~~π~~ $L := \pi^*(K_X) + \Delta = K_{X^n} + \Delta^n + \Gamma$ is semi-ample on X^n , we get fibre space $\underline{g^n : X^n \rightarrow Y^n}$. Let $h^n : \Gamma^n \rightarrow T^n$ be the fibre space induced by $\underline{|mL|_{\Gamma^n}}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \Gamma^n & \xrightarrow{\tau_1} & X^n \\ \text{fibre } \downarrow h^n & & \downarrow g^n \text{ fibre } f \\ T^n & \xrightarrow{\sigma_1} & Y^n \end{array}$$



Where the morphisms $\tau_1, \tau_2; \Gamma^n \rightarrow X^n$ induce morphisms $\sigma_1, \sigma_2: T^n \rightarrow Y^n$. Where $\underline{g^n}: (X^n, \Delta^n + \Gamma) \rightarrow Y^n$ and $h^n: (\Gamma^n, \underline{\Theta}) \rightarrow T^n$ give minimal qlc structures, which induce minimal qlc-stratifications. Where $K_{\Gamma^n} + \underline{\Theta} = (K_{X^n} + \Delta^n + \Gamma)|_{\Gamma^n}$

$K_{X^n} + \Delta^n + \Gamma$ is triv.

Theorem 15 (3.13)

The quotient Y of $T^n \rightrightarrows Y$ exists. Furthermore, there exists a morphism $g : X \rightarrow \underline{Y}$.

Proof

Our aim is to use Theorem (3.8), we need to prove it is a stratified, finite, set theoretic equivalence relation.

(HN) and (HSN) follow
from having "f"-glc stratified

Lemma 16 (3.11)

$\sigma : T \rightrightarrows Y$ gives a stratified equivalence relation.

Proof

Proof of Lemma 3.11

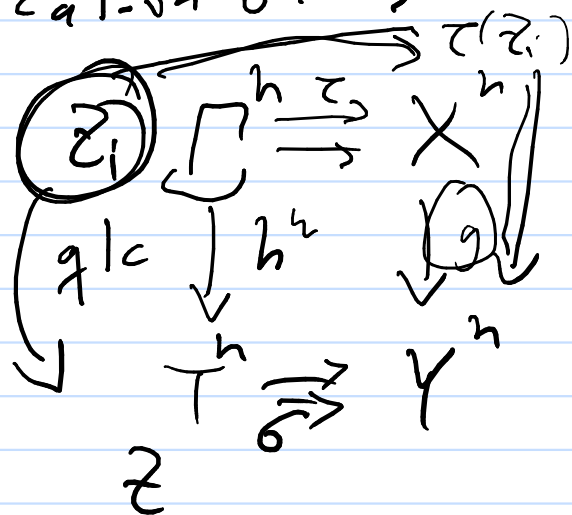
We want to show. for σ_j

$\sigma_j^{-1}S_i$ coincide with the stratification on T_i

We want $S_i: T^n = \sigma_j^{-1}(S_i: Y^n)$

$S_i: T^n \subseteq \sigma_j^{-1}(S_i: Y^n)$, by ind.

\bar{Z} is in strata of T^n .



$h^n(Z_i) = \bar{Z}$.

$\tau(Z_i)$ is also non klt center of $(X^n, \Delta^n + \Gamma)$

$\Rightarrow Z_i$

$\sigma_j(\bar{Z}) = g^n(\underbrace{\tau(Z_i)}_{\subseteq S_i^* X^n}) \subseteq S_i^* Y^n$ as g^n is strat.

$\Rightarrow \sigma_j(\bar{Z}) = \sigma_j(\bar{Z}) \setminus S_{i-1}^*(T^n)$

$\subseteq S_i^*(Y^n) \setminus S_{i-1}^* Y^n = S_i: Y^n$

by ind

Lemma 17 (3.12)

$\sigma : T \rightrightarrows Y$ generates a finite set theoretic equivalence relation.

Proof

Proof of Lemma 3.12

By Lemma (3.9) we only need to check

for $R_{X/Z}$. For "special" Z ,

Special $Z \simeq \boxed{Y/U \cup U^0}$ we check on
 U^0 .

Here $(K_X + \Delta)|_{X^0}$ over U^0 semi-ample.

over U^0 Y^0 is the quotient of
 Y^n/U^0 Y^n/T^n over U^0

\hookrightarrow Finiteness over U^0

\hookrightarrow Finiteness over U .

Proof of main Proposition

Proposition 1 (3.1)

Let (X, Δ) be a dslt pair, projective over a normal variety U . And $n : X^n \rightarrow X$ be the normalization. Write

$n^*(K_X + \Delta) = K_{X^n} + \Delta^n + \Gamma$ Where Γ is the double-locus.
Assume that:

- There exists an open set $U^0 \subset U$, such that if we write $(X^0, \Delta^0) = (X, \Delta) \times_U U^0$, then $K_{X^0} + \Delta^0$ is semi-ample over U^0 .
- The image of any non-klt center of $(X^n, \Delta^n + \Gamma)$ intersects U^0 , and
- $K_{X^n} + \Delta^n + \Gamma$ is semi-ample over U^0 .

Then, $K_X + \Delta$ is semi-ample over U .

Technical Result

Theorem 18 (Gongyo 10)

For (X, Δ) a klt pair, with $K_X + \Delta \sim_{\mathbb{Q}} 0$. The image of $\rho_m : (\text{Bir}(X, \Delta)) \rightarrow \text{Aut}(H^0(X, m(K_X + \Delta)))$ is a finite group for a sufficiently large and divisible m .

Proof (of Proposition 3.1)

Proof of 3.1

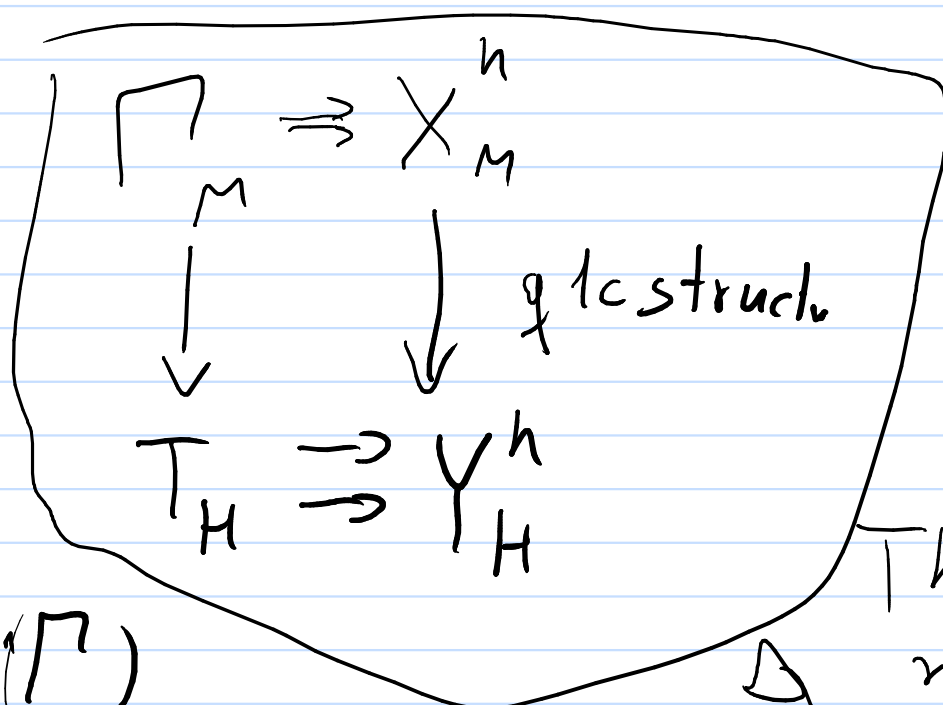
Take m . $M := m(K_{X^n} + \Delta^n + \Gamma)$
 Cartier and bpf.

$\tilde{g}: X^n \rightarrow Y^n$ f.b. over U .
 with very ample H on Y^n s.t.

$$g^{n*}(\underline{H}) = \underline{M}$$

$$\rho_x: X^n \rightarrow X^n$$

$$\rho_y: Y^n \rightarrow Y^n$$



$$\Delta_M^n := \rho_x^{-1}(\Delta^n) \text{ and } \Gamma_M^n := \rho_x^{-1}(\Gamma)$$

$$(X_M^n, \Delta_M^n + \Gamma_M^n) \rightarrow Y_H^n \text{ mir. glc}$$

This map makes sense

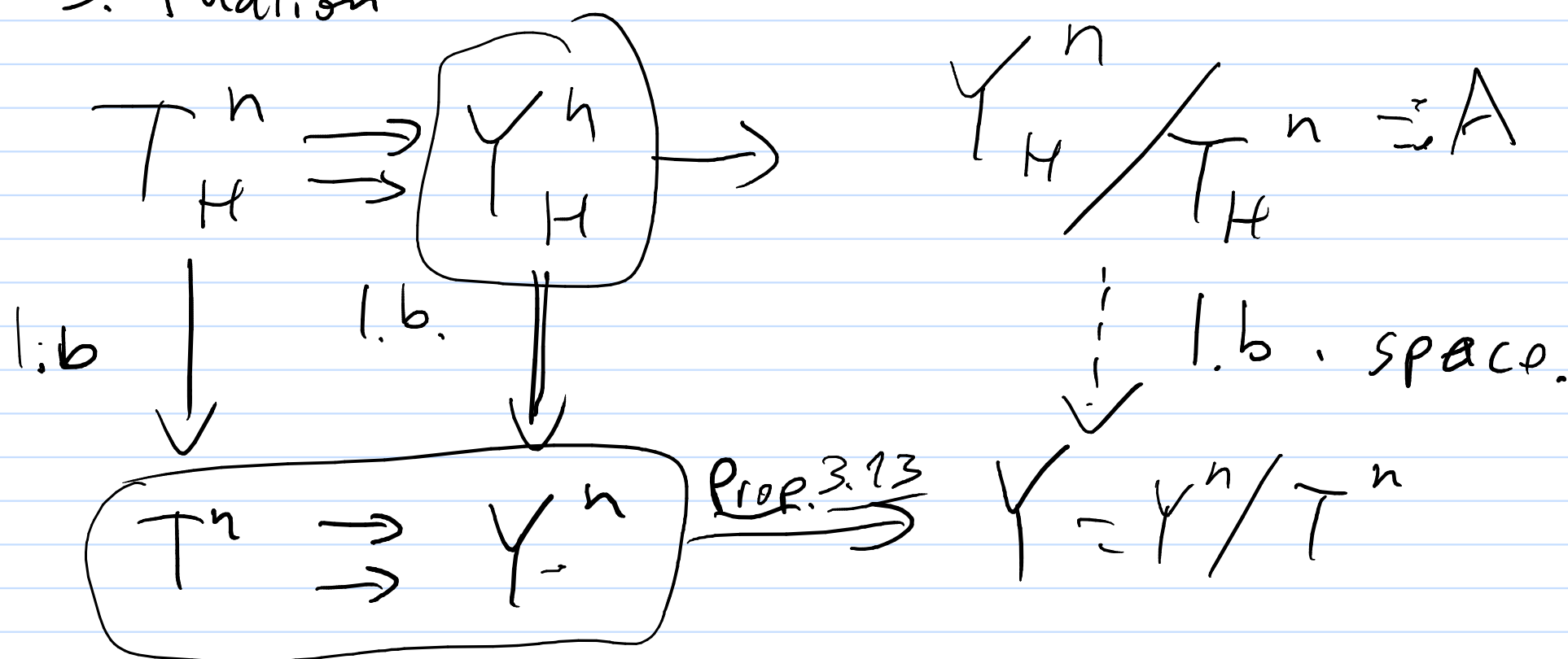
$$\Gamma_M^n \text{ total spaces of } M|_H \text{ } H|_{T_H^n}$$

$$\sigma_{1H} \sigma_{2H}: T_H^n \rightarrow Y_H^n$$

we need to check conditions for the quotient to exist.

...

Situation



A is a line bundle over $Y = Y^n / T^n$
 with pull-back to Y^n Y_H^n .

$$\mathcal{O}_{Y^n}(\underline{H}).$$

H is v. ample over U .

$\Rightarrow A$ is ample over U .

$$\mathcal{O}_x(\underbrace{m(K_x + \Delta)}_M) = g^*(A)$$

$m(K_x + \Delta)$ is semi-ample over U

□

Theorem 19 (4.1)

Let $f : X \rightarrow U$ be a projective morphism and (X, Δ) be a \mathbb{Q} -factorial dlt pair. Assume that there exists an open subset $U^0 \subset U$, such that:

- the image of any strata S_i of $S = \lfloor \Delta \rfloor$ intersects U^0
- $K_X + \Delta$ is nef and $(K_X + \Delta)|_{X^0}$ is semi-ample over U^0 , where $X^0 = X \times_U U^0$, and
- for any component S_i of S , $(K_X + \Delta)|_{S_i}$ is semi-ample over U .

Then $K_X + \Delta$ is semi-ample over U .

Proof

F i n

F i n

F i n

F in

Theorem 20 (Fujino)

Let (X, Δ) be an lc pair and let $f : X \rightarrow U$ be a proper morphism onto a variety U . Assume the following conditions:

- H is a f -net \mathbb{Q} -Cartier \mathbb{Q} -divisor on X .
- $H - K_X + \Delta$ is f -nef and f -abundant.
- $\kappa(X_\eta, (aH - (K_X + \Delta))_\eta) \geq 0$ and $\nu(X_\eta, (aH - (K_X + \Delta))_\eta) = \nu(X_\eta, (H - (K_X + \Delta))_\eta)$ for some $1 < a \in \mathbb{Q}$. Where η is the generic point of U .
- There is a positive integer c such that cH is Cartier and that $\mathcal{O}_T(cH) := \mathcal{O}_X(cH)|_T$ is f -generated, where T is the non-klt locus of (X, Δ)