Existence of log canonical closures

Minimal Model Program Learning Seminar

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Main Proposition

Proposition 1 (3.1)

Let (X, Δ) be a dslt pair, projective over a normal variety U. And $n : X^n \to X$ be the normalization. Write $n^*(K_X + \Delta) = K_{X^n} + \Delta^n + \Gamma$. Where Γ is the double-locus. Assume that:

- *There exists an open set $U^0 \subset U$, such that if we write $(X^0, \Delta^0) = (X, \Delta) \times_U U^0$, then $K_{X^0} + \Delta^0$ is semi-ample over U^0.*
- *The image of any non-klt center of $(X^n, \Delta^n + \Gamma)$ intersects U^0, and*
- *$K_{X^n} + \Delta^n + \Gamma$ is semi-ample over U^0.*

Then, $K_X + \Delta$ is semi-ample over U.

Definition 1 (3.2)

Let X be a scheme. A *stratification* of X is a decomposition of X into a finite disjoint union of reduced locally closed sub-schemes We write $X = \cup_i S_i X$ where $S_i X_i \subset X$ is the i-th dimensional stratum. Such a stratified scheme is denoted by (X, S_*). We assume that $\boxed{\cup_{i \leq j} S_i X}$ is closed for every j.
The boundary of (X, S_*) is the closed subscheme

$$\underline{BX := \cup_{i<dimX} S_i X} = X \backslash S_{dimX} X.$$

Let (X, S_*), (Y, S_*) be stratified schemes. We say that $f : X \to Y$ is a *stratified morphism* if $f(S_i X) \subset S_i Y$ for every i.

$\cup_{i \leq j} S_i X =: S_i^+ X$ is also closed

Example 2

Let (X, Δ) be a log canonical pair. Let $S_i^* \subset X$ be the union of all non-klt centers of (X, Δ) of dimension $\leq i$, and $S_i X := S_i^*(X, \Delta) \setminus S_{i-1}^*(X, \Delta)$. We call this the *lc stratification* of $(X, /Delta)$.

(X, Δ)

Definition 3 (3.4)

(N) We say that (X, S_*) has *normal strata* if each S_iX is normal

(SN) We say that (X, S_*) has *seminormal boundary* if X and the boundary BX are both seminormal

(HN) We say that (X, S_*) has *hereditary normal boudary* if:
- X satisfies (N)
- The normalization $\pi : X^n \to X$ is stratifiable, and
- $B(X^n)$ satisfies (HN)

$\emptyset$

Sat. HN
HSN

(HSN) We say that (X, S_*) has *hereditary seminormal boundary* if:
- X satisfies (SN)
- The normalization $\pi : X^n \to X$ is stratifiable, and
- $B(X^n)$ satisfies (HSN)

We use HN and HSN to get
Quotients by finite relations

The lc stratification satisfies (N), (SN),(HN) and (HSN).

Example 4

Take

$$X = (x^2 = y^2(y + z^2)) \subset (A)^3$$

. With $S_1 = (x = y = 0)$. Then S_1 and S_2 are smooth. The normalzation of X is:

$$X^n = (x_1^2 = y + z^2) \subset \mathbb{A}^3.$$

$x_1 = \frac{x}{y}$

And the preimage of $S_1 X$ is $(y = x_1^2 - z^2 = 0)$, which is not normal.

So it has (N) but not (HN).

Definition 5 (3.6)

Let Y be a normal scheme. A *minimal qlc structure* on Y is a proper surjective morphism $f : (X, \Delta) \to Y$, where:

- (X, Δ) is a log canonical pair
- $\mathcal{O}_Y = f_*\mathcal{O}_X$, and
- $K_X + \Delta \sim_{f,\mathbb{Q}} 0$

quasi log canonical.

Definition 6 (3.7)

Let $f : (X, \Delta) \to Y$ be a minimal qlc structure. We define the *f-qlc stratification* $(Y, S_*(X/Y, \Delta))$ in the following way. Let $\mathcal{H}_X$ denote the set of all non-klt centers of (X, Δ). For each $Z \in \mathcal{H}_X$, let:

$$W_Z = f(Z) \setminus \bigcup_{Z' \in (H)_X, f(Z) \not\subseteq f(Z')} f(Z').$$

Then $Y = \amalg_{Z \in \mathcal{H}_X} W_Z$ is the *qlc-stratification*.

$\cup W_Z$ of given dimension give the stratification

qlc strat. satisfies (HN) an (HSN)

Definition 7

Let X and R be U-schemes. A pair of morphisms $\sigma_1, \sigma_2 : R \rightrightarrows X$ is called a *pre-relation*. It is called *finite* if both morphisms are finite and a *relation* if $\sigma : R \to X \times_U X$ is a closed embedding.

Definition 8 (3.3)

Let $(X < S_*)$ be a stratified scheme. A relation $\sigma : R \rightrightarrows X$ is *stratified* if each σ_i is stratifiable and $\sigma_1^{-1} S_i = \sigma_2^{-1} S_i$.

pre-relation $\rightsquigarrow$ relation ✓

Definition 9

Let X and R be reduced U-schemes. We say that a relation $\sigma : \underbrace{R \rightrightarrows X}$ is a *set theoretic equivalence relation* if:

- σ is geometrically injective.
- R contains the diagonal. (reflexive)
- There is an involution τ_R on R, such that $\underbrace{\tau_{X\times X}} \circ \sigma \circ \tau_R = \sigma$. (symmetric)
- For $1 \leq i < j \leq 3$. Taking $X_i := X$ and $R_{ij} \subset X_i \times_U X_j$. Then the coordinate projection of $\underbrace{red}(R_{12} \times_{X_2} R_{23})$ to $X_1 \times_U X_2$ 3 factors:

$$red(R_{12} \times_{X_2} R_{23}) \to R_{13} \to X_1 \times X_3.$$

R_{13}

transitive.

$\boxed{\text{Fin. pre-rel, eq.}} \rightsquigarrow \boxed{\text{pro-f. rel, eq.}}$

Definition 10

Let $\sigma : R \rightrightarrows X$ be a set theoretic equivalence relation. We say that $q : X \to Y$ is a *geometric quotient* of X by R if:

- $q \circ \sigma_1 = q \circ \sigma_2$.
- $q : X \to Y$ is universal with this property.
- $q : X \to Y$ is finite.

The geometric quotient is denoted by X/R.

$Y = X/R$

$$\begin{tikzcd} & Y \arrow[d, dashed, "\exists !"] \\ X \arrow[ur] \arrow[r] & Z \end{tikzcd}$$

Commuting.

Lemma 11

Let $R \rightrightarrows X$ be a finite, set theoretic equivalence relation with X, R reduced and over a field of characteristic 0. let $\pi : X' \to X$ and $q' : X' \to Z$ be finite surjections, with either

- *X, Z are semi-normal and the geometric fibers of q' are exactly the pre-images of R-equivalence classes.*
- *X, Z normal and such that the $\sigma_i : R \to X$ are open and over a dense subset of Z, the geometric fibers of q' are exactly the pre-images of R-equivalence classes.*

Then $Z = X/R$

$$\begin{tikzcd}
 & & X' \arrow[dl, "p"', "\text{fin.}"] \arrow[dr, "q"] & \\
R \arrow[r, shift left] \arrow[r, shift right] & X & & \boxed{Z}
\end{tikzcd}$$

Theorem 12 (3.8)

Let (X, S_*) be a stratified excellent scheme or algebraic space over a field of characteristic 0. Assume that (X, S_*) satsfies (HN) and (HSN). Let $R \rightrightarrows X$ be a finite,stratified, set theoretic equivalence relation. Then:

- *The geometric quotient X/R exists.*
- *$\pi : X \to X/R$ is stratifiable, and*
- *$(X/R, \pi_* S_*)$ also satisfies (HN) and (HSN).*

Proof

Lemma 13

Let (X) be an excellent scheme over a field of characteristic 0 that is normal and of pure dimension d. Let $R \rightrightarrows X$ be a finite, set theoretic equivalence relation. Let $R^d \subset R$ denote the $d-$dimensional part of R. Then

- *$R^d \rightrightarrows X$ is a finite, set theoretic equivalence relation,*
- *The geometric quotient $\boxed{X/R^d}$ exists, and*
- *X/R^d is normal.*

Proof

Idea is to get some

$S \to$ "X" and "X"$/S \simeq X/R^d$

Proof of Lemma 13

$R^d \rightrightarrows X$ reflexive, symmetry. Follow from $R \rightrightarrows X$.
Transitive property does not.

$\sigma_i : R^d \to \underline{X}$ Finite morphism, image is normal, and both pure dimension

$\Rightarrow$ σ_i is universally open.

$\underbrace{R^d \times_X R^d}_{\text{has pure dimension } d.} \to \boxed{R^d}$ is open.

Therefore, $R^d \times_X R^d \to R$ lies in the d-dim part R.

$R^d \times R^d \to R \times R \to R \to X \times X$

$\underbrace{R^d \times R^d \to R^d \subseteq R \to X \times X}_{\text{splitting of projections.}}$

Construct the quotient

we can assume X irreducible.

$\underbrace{X \times \dots \times X}_{\deg \sigma_1}$ with $\pi_i : X \times \dots \times X \to X$ each coord proj.

And $\underbrace{R_{ij}}$, Δ_{ij} $(\pi_i \times \pi_j)^{-1}(R)$
$(\pi_i \times \pi_j)^{-1}(\Delta)$

Geometric points $\bigcap_{ij} R_{ij}$ $(x_1, \dots, x_m)$

s.t. any 2 are R-equivalent.

$\bigcap R_{ij} \setminus \bigcup \Delta_{ij}$ sequences $(x_i, \dots, x_n)$
giving entire equivalence classes.

$\tilde{X}' \xrightarrow{n} \overline{\underbrace{\bigcap R_{ij}} \setminus \bigcup D_{ij}}$.

$S_m \curvearrowright X \times \dots \times X$.
$\curvearrowright \bigcap R_{ij} \setminus \bigcup D_{ij}$

lift $S_m \curvearrowright \boxed{X'}$.

over a dense subset of X the S_m orbits are R-equivalence class.

$$\begin{tikzcd} & X' \arrow[dl] \arrow[dr] & \\ \underline{X} & & \underline{X'/S_m} \end{tikzcd}$$

criterion,
$\underbrace{X'/S_m}_{\substack{\text{quot. by fin. group.} \\ \exists}} \doteq \underline{X/R}_{\exists.}$

$Y \xrightarrow{n} X/R^d$, X is normal

$$\begin{tikzcd} & & Y \arrow[dr, "n"] & \\ R \arrow[r, shift left, "\sigma"] \arrow[r, shift right] & X \arrow[ur, "\text{has to com with } \sigma_1, \sigma_2"] \arrow[rr] \arrow[dr] & & X/R^d \arrow[dl, "\exists!"] \\ & & W & \end{tikzcd}$$

lift, lifts

$\Rightarrow$ its the g. quotient.

Proof of Theorem 3.8

$$R \rightrightarrows X$$

We use induction $d = \dim X$.

$$\begin{tikzcd} (X^n, S^n) \arrow[r] & (X^n, S_*) \\ R^n \arrow[u, shift left] \arrow[u, shift right] & R \arrow[u, shift left] \arrow[u, shift right] \end{tikzcd}$$

pullback. keeps conditions on R.

$$X^{nd} \subseteq X^n \qquad d\text{-dim. irr. comp.}$$

$$X^{nd} \to X^{nd}/R^{nd} \text{ exists and normal.}$$

$$X^{nl} := X \setminus X^{nd} \qquad \text{the lower dim. part.}$$

$$R^{nd} \rightrightarrows X^n \qquad \text{(by being trivial in } X^{nl}\text{)}$$

$$X^n/R^{nd} = (X^{nd}/R^{nd}) \sqcup X^{nl}.$$

$$\Rightarrow B(X^n/R^{nd}) = \underbrace{B(X^{nd}/R^{nd})}_{\substack{\text{quotient in boundaries} \\ \text{exist by induction.}}} \sqcup X^{nl}.$$

$$\begin{tikzcd} B(X^{nd}/R^{nd}) \sqcup X^{nl} = B(X^n/R^{nd}) \arrow[r, hook] \arrow[d] & X^n/R^{nd} \arrow[d] \\ B(X^n/R^{nd})/_{q.} R^{n\prime\prime} \arrow[r] & X^n/_{q.} R^{n\prime\prime} \end{tikzcd}$$

Y will be the quotient $X^n/R^n \quad = R^n$.

$\square$

Lemma 14 (3.9)

Let (X, S_*) be a stratified space satisfying (N) and $Z \subset X$ a closed subspace which does not contain any of the irreducible components of the $S_i X$. Let $R \rightrightarrows (X, S_*)$ be a pro-finite, stratified, set theoretic equivalence relation. If $R|_{X \setminus Z}$ is a finite set theoretic equivalence relation, then R is also a finite set theoretic equivalence relation.

Proof

pro-finite

R is a union of finite relations

We need to check that R has finitely many components

Proof of Lemma 3.9

We can do this Strata by Strata.

we have normal.

Every irred comp of R dominates an irreducible component of X.

Finiteness $\boxed{\text{over a dense subset}}$ $\boxed{X \setminus Z}$ gives finitely many components of R.

$\square$

Let (X, Δ) be a dslt pair, Γ^n the normalization of the double locus of $\Gamma \subset X^n$ and $\tau : \Gamma^n \to \Gamma^n$ the induced involution. Then the relation $\tau_1, \tau_2 : \Gamma^n \rightrightarrows X^n$ has quotient given by the normalization:

$$\pi : X^n \to X = X^n/\Gamma^n.$$

~~As~~ $L := \pi^*(K_X) + \Delta = K_{X^n} + \Delta^n + \Gamma$ is semi-ample on X^n, we get fibre space $g^n : X^n \to Y^n$. Let $h^n : \Gamma^n \to T^n$ be the fibre space induced by $|mL|_{\Gamma^n}|$. Then we have the following commutative diagram:

$$\begin{tikzcd}
\Gamma^n \arrow[r, shift left, "\tau_i"] \arrow[r, shift right] \arrow[d, "h^n", "\text{fibre}"'] & X^n \arrow[d, "g^n"', "\text{fibre}"] \\
T^n \arrow[r, shift left] \arrow[r, shift right, "\sigma_i"'] & Y^n
\end{tikzcd}$$

$$
\begin{tikzcd}
 & \Gamma^n \arrow[r, shift left, "\tau"] \arrow[r, shift right] \arrow[d, "h^n"] & X^n \arrow[r] \arrow[d, "g^n"'] & X = X^n/\Gamma^n \\
\text{qlc strata} & \boxed{T^n} \arrow[r, shift left] \arrow[r, shift right] & \boxed{Y^n} \arrow[r] & Y = Y^n/\Gamma^n
\end{tikzcd}
$$

qlc

Where the morphisms $\tau_1, \tau_2; \Gamma^n \to X^n$ induce morphisms $\sigma_1, \sigma_2 : T^n \to Y^n$. Where $g^n : (X^n, \Delta^n + \Gamma) \to Y^n$ and $h^n : (\Gamma^n, \Theta) \to T^n$ give minimal qlc structures, which induce minimal qlc-stratifications. Where $K_{\Gamma^n} + \Theta = (K_{X^n} + \Delta^n + \Gamma)|_{\Gamma^n}$

$K_{X^n} + \Delta^n + \Gamma$ is triv.

Theorem 15 (3.13)

The quotient Y of $T^n \rightrightarrows Y$ exists. Furthermore, there exists a morphism $g : X \to Y$.

Proof

Our aim is to use Theorem (3.8), we need to prove it is a stratified, finite, set theoretic equivalence relation.

(HN) and (HSN) follow from having "F'-glc stratificat

Lemma 16 (3.11)

$\sigma : T \rightrightarrows Y$ *gives a stratified equivalence relation.*

Proof

Proof of Lemma 3.11

We want to show. For σ_1

$\sigma_j^{-1} S_i$ coincide with the stratification on T.

We want $\boxed{S_i T^n = \sigma_j^{-1}(S_i Y^n)}$

$S_i T^n \subseteq \sigma_j^{-1}(S_i Y^n)$. by ind

Z i-dim strata of T^n.

$$\begin{tikzcd}
Z_i \arrow[d, bend right, "g|_c"'] \arrow[rr, bend left, "\tau(Z_i)"] & \Gamma^n \arrow[r, "h\ \tau"] \arrow[r, shift right] \arrow[d, "h^n"] & X^n \arrow[d, "g"] \\
Z & T^n \arrow[r, "\sigma"'] \arrow[r, shift left] & Y^n
\end{tikzcd}$$

$h^n(Z_i) = \overline{Z}$.

$\tau(Z_i)$ is also non klt center of $(X^n, \Delta^n + \Gamma)$

$\Rightarrow Z_i$

$\sigma_1(\overline{Z}) = g^n(\underbrace{\tau(Z_i)}_{\subseteq S_i^* X^n}) \subseteq S_i^* Y^n$ as g^n is strat.

$\Rightarrow \sigma_i(Z) = \underbrace{\sigma_j(\overline{Z})}_{} \setminus \boxed{S^*_{i-1}(T^n)}$

$\subseteq S_i^*(Y^n) \setminus S^*_{i-1} Y^n = S_i Y^n$

by ind

Lemma 17 (3.12)

$\sigma : T \rightrightarrows Y$ generates a finite set theoretic equivalence relation.

Proof

Proof of Lemma 3.12

By Lemma (3.9) we only need to check for $R_{X/Z}$. for "special" Z.

Special $Z \simeq \boxed{Y|_{U \setminus U^\circ}}$. we check on $\underline{U^\circ}$.

Here $(K_X+\Delta)|_{X^\circ}$ over U° semi-ample.

over $\underline{U^\circ}$ Y° is the quotient of

$Y|_{U^\circ}^{n}$ " Y^n / T^n over U°

$\Leftrightarrow$ finiteness over U°

$\Leftrightarrow$ finiteness over U.

Proof of main Proposition

Proposition 1 (3.1)

Let (X, Δ) be a dslt pair, projective over a normal variety U. And $n : X^n \to X$ be the normalization. Write
$\boxed{n^*(K_X + \Delta) = K_{X^n} + \Delta^n + \Gamma}$ *Where Γ is the double-locus.*
Assume that:

- *There exists an open set $U^0 \subset U$, such that if we write $(X^0, \Delta^0) = (X, \Delta) \times_U U^0$, then $K_{X^0} + \Delta^0$ is semi-ample over U^0.*
- *The image of any non-klt center of $(X^n, \Delta^n + \Gamma)$ intersects U^0, and*
- *$K_{X^n} + \Delta^n + \Gamma$ is semi-ample over U^0.*

Then, $K_X + \Delta$ is semi-ample over U.

Technical Result

Theorem 18 (Gongyo 10)

For (X, Δ) a klt pair, with $K_X + \Delta \sim_{\mathbb{Q}} 0$. The image of $\rho_m : (Bir(X, \Delta)) \to Aut(H^0(X, m(K_X + \Delta)))$ is a finite group for a sufficiently large and divisible m.

Proof (of Proposition 3.1)

Proof of 3.1

Take m. $\quad M := m(\underbrace{K_{X^n}+\Delta^n+\Gamma}_{\text{Cartier and bpf.}})$

$g: X^n \to Y^n$ f.b. over U.

with very ample H on Y^n s.t.

$g^{n*}(\underline{H}) = \underline{\underline{M}}$.

$p_x: X_M^n \to X^n$

$p_y: Y_H^n \to Y^n$

$$\begin{tikzcd} \Gamma_M \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X_M^n \arrow[d, "\text{glc struct.}"] \\ T_H \arrow[r, shift left] \arrow[r, shift right] & Y_H^n \end{tikzcd}$$

This map makes sense

$\Delta_M^n := p_x^{-1}(\Delta^n)$ and $\Gamma_M := p_x^{-1}(\Gamma)$

$(X_M^n, \Delta_M^n + \Gamma_M) \to Y_H^n$ min. glc

Γ_M^n totalspases of $M|_{\Gamma^n}$ $H|_{T^n}$.

T_H^n

$\sigma_{1H}, \sigma_{2H}: T_H^n \rightrightarrows Y_H^n$.

we need to check conditions for the quotient to exist.

...

Situation

$$\begin{tikzcd} T_H^n \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\text{l.b}"'] & \boxed{Y_H^n} \arrow[r] \arrow[d, "\text{l.b.}"'] & Y_H^n / T_H^n =: A \arrow[d, dashed, "\text{l.b. space.}"] \\ T^n \arrow[r, shift left] \arrow[r, shift right] & Y^n \arrow[r, "\text{Prop. 3.13}"] & Y = Y^n/T^n \end{tikzcd}$$

A is a line bundle ove $Y = Y^n/T^n$

with pull-back to Y^n $\quad Y_H^n$.

$\mathcal{O}_{Y^n}(H)$.

H is v. ample over. U.

$\Rightarrow A$ is ample over U.

$\mathcal{O}_X(\underbrace{m(K_X+\Delta)}_{M}) = g^*(A)$

$m(K_X+\Delta)$ is semi-ample over U.

$\square$

Theorem 19 (4.1)

Let $f : X \to U$ be a projective morphism and (X, Δ) be a $\mathbb{Q}$-factorial dlt pair. Assume that there exists an open subset $U^0 \subset U$, such that:

- *the image of any strata S_i of $S = \lfloor \Delta \rfloor$ intersects U^0*
- *$K_X + \Delta$ is nef and $(K_X + \Delta)|_{X^0}$ is semi-ample over U^0, where $X^0 = X \times_U U^0$, and*
- *for any component S_i of S, $(K_X + \Delta)|_{S_i}$ is semi-ample over U.*

Then $K_X + \Delta$ is semi-ample over U.

Proof

F_{in}

F_{in}

Fin

F in

Theorem 20 (Fujino)

Let (X, Δ) be an lc pair and let $f : X \to U$ be a proper morphism onto a variety U. Assume the following conditions:

- *H is a f-net $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X.*
- *$H - K_X + \Delta$ is f-nef and f−abundant.*
- *$\kappa(X_\eta, (aH - (K_X + \Delta))_\eta) \geq 0$ and $\nu(X_\eta, (aH - (K_X + \Delta))_\eta) = \nu(X_\eta, (H - (K_X + \Delta))_\eta)$ for some $1 < a \in \mathbb{Q}$. Where η is the generic point of U.*
- *There is a positive integer c such taht cH is Cartier and that $\mathcal{O}_T(cH) := \mathcal{O}_X(cH)|_T$ is f-generated, where T is the non-klt locus of (X, Δ)*